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VORTICAL MOMENTUM OF FLOWS OF AN INCOMPRESSIBLE LIQUID

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UDC 532.51

§1. In an incompressible liquid, filling the whole space outside of the finite region K (with a sufficiently smooth closed surface ∂K), let there be given a field of the velocities $\mathbf{v}(\mathbf{r}, t)$, satisfying the following conditions: a) sufficient smoothness; b) div $\mathbf{v}=0$; c) $|\mathbf{v}(\mathbf{r})| \sim \text{const}/r^{1+\varepsilon}$, $|\text{rot } \mathbf{v}| \sim \text{const}/r^{4+\varepsilon}$ with $r \equiv |\mathbf{r}| \to \infty$ with small $\varepsilon > 0$.

We denote by $\mathbf{n}$ the external normal to ∂K, and by G the region filled with the liquid. Let the liquid density $\rho = 1$. It is convenient to use the following representation for $\mathbf{v}(\mathbf{r}, t)$;

$$\mathbf{v}(\mathbf{r}) = \text{grad } \varphi + \text{rot } \mathbf{A}; \tag{1.1}$$

$$\begin{cases} \varphi(\mathbf{r}) = -\dfrac{1}{4\pi} \oint\limits_{\partial K} \dfrac{\hat{\mathbf{n}} \cdot \hat{\mathbf{v}}}{s} dS, \\ \mathbf{A}(\mathbf{r}) = \dfrac{1}{4\pi} \left[\int\limits_G \dfrac{\hat{\boldsymbol{\omega}}}{s} dV + \oint\limits_{\partial K} \dfrac{\hat{\mathbf{n}} \times \hat{\mathbf{v}}}{s} dS \right], \end{cases} \tag{1.2}$$

where $s = |\mathbf{r} - \hat{\mathbf{r}}|$; the symbol "$\wedge$" denotes variables with respect to which integration is carried out; $\omega \equiv$ rot $\mathbf{v}$.

This representation for the case of a finite G is given in [1]; for an infinite region G it is proved by a direct calculation [substitution of (1.2) into (1.1)], taking account of limitations on the asymptotic of the field of the velocity. Here a formal calculation gives $\mathbf{v}(\mathbf{r}) \equiv 0$ in the region K outside the liquid.

The latter shows that the flow in G can be integrated as the flow in the whole space, obtained by "filling" of the region K with a liquid at rest. Under these circumstances, at ∂K there is a discontinuity of both the tangential and normal components of the velocity, corresponding to the distribution (1.2) of the vortices of the density $\mathbf{n} \times \mathbf{v}$ and sources of the density $\mathbf{n} \cdot \mathbf{v}$ at ∂K.

Naturally, the region K can be filled in any other arbitrary way (not necessarily by a liquid at rest); under these circumstances, there is a change in the distribution of the vortices and sources in (1.2).

However, the representation (1.2) has the advantage that the "filling" of K with a liquid at rest does not change the total momentum of the flow, which will be important in what follows with a generalization of the concept of momentum.

§2. Momentum of Flows of an Incompressible Liquid. The usual definition of the momentum of a flow, which we shall call the "true" momentum I, has the form

$$\mathbf{I} = \int\limits_{V_0} \mathbf{v} \, dV, \tag{2.1}$$

where V_0 is the volume occupied by the liquid; $\mathbf{v} = \mathbf{v}(\mathbf{r}, t)$ is the field of the velocity.

This definition is applicable for both finite and infinite V_0, with the condition of the absolute convergence of the integral (2.1).

However, in the case of a liquid filling the whole space outside some limited system of bodies, the integral (2.1), for flows having a dipolar asymptotic, does not converge absolutely. Its value is found to depend on the manner in which the volume of the integration approaches infinity. In the case of absolute convergence of (2.1) for a liquid filling the whole space, $\mathbf{I}=0$; i.e., for all such flows with a zero momentum, the definition (2.1) has no meaning.

Novosibirsk. Translated from Zhurnal Prikladnoi Mekhaniki i Tekhnicheskoi Fiziki, No. 6, pp. 72-77, November-December, 1977. Original article submitted November 19, 1976.

Therefore, for the case of a liquid filling the whole space, the vortical momentum was introduced [2]:

$$\mathbf{P} = \frac{1}{2}\int \mathbf{r} \times \boldsymbol{\omega} dV. \tag{2.2}$$

It is well known that the vector P, having the dimensions of momentum, is conserved for flows of both viscous (Stokes) and ideal incompressible liquids in the absence of nonpotential forces [3].

Let us now examine a flow of the type described in Sec. 1. In a concrete realization of this flow, the region K can be either a cavern (cavity) or a solid or deformed body. In accordance with the method adopted in Sec. 1 for reducing the flow under consideration to a flow in the whole space, without a change in the total momentum of the flow, we postulate that, to determine the vortical momentum, it is sufficient to use in (2.2) the field of the vortex from (1.2). This field consists of the distribution of ω in G and the surface distribution of the density of $\mathbf{n} \times \mathbf{v}$ at ∂K.

Then the definition of the vortical momentum assumes the form

$$\mathbf{P} = \frac{1}{2}\left\{\int_G \mathbf{r} \times \boldsymbol{\omega} dV + \oint_{\partial K} \mathbf{r} \times (\mathbf{n} \times \mathbf{v})\, dS\right\}. \tag{2.3}$$

We shall prove the correctness of this definition by a theorem.

<u>THEOREM 1</u>. In the region G let there be given a flow of an incompressible viscous (obeying the Navier–Stokes equations) liquid, such that the region K and the field of the velocities $\mathbf{v}(\mathbf{r}, t)$ satisfy the conditions formulated in Sec. 1. In addition, let the motion take place in a field of gravity $g = \text{const}$. Then, for the vortical momentum (2.3) we have

$$\frac{dP_i}{dt} = -g_i V - \oint_{\partial K} \sigma_{ik} n_k dS, \tag{2.4}$$

where $\sigma_{ik} = -p\delta_{ik} + \nu(\partial v_i/\partial x_k + \partial v_k/\partial x_i)$; V is the volume of the region K; ν is the coefficient of kinematic viscosity; p is the pressure. Here we use the summation law for repeated indices. The theorem is valid for both viscous and ideal liquids.

<u>Proof.</u> In tensor form (2.3) assumes the form

$$2P_i = \int_G \varepsilon_{ikl} x_k \omega_l dV + \int_{\partial K} \varepsilon_{ikl}\varepsilon_{lmn} x_k v_n n_m dS, \tag{2.5}$$

where ε_{ikl} is a unit antisymmetrical tensor of the third rank.

We take the Navier–Stokes equation in the form

$$\begin{aligned} \frac{dv_n}{dt} &= -\frac{\partial p}{\partial x_n} + g_n + \nu \frac{\partial^2 v_n}{\partial x_a \partial x_a}, \\ \frac{d\omega_l}{dt} &= -\varepsilon_{lmn}\frac{\partial v_a}{\partial x_m}\frac{\partial v_n}{\partial x_a} + \nu\frac{\partial^2 \omega_l}{\partial x_a \partial x_a}. \end{aligned} \tag{2.6}$$

We differentiate (2.5) with respect to the time, taking into consideration that G is the liquid volume, ∂K is the liquid surface. Under these circumstances, we use the well known rules of differentiation with respect to liquid configurations [4] and Eqs. (2.6). With a transition from surface integrals to volumetric integrals, and the reverse, we use the bounded character of the asymptotic of the field of the velocity, shown in Sec. 1. After rather cumbersome transformations, we obtain (2.4), where the first term on the right-hand side is the Archimedes force and the second the force acting from the side of the region K on the liquid surrounding it. If the system is at rest, then $dP_i/dt = 0$, and we obtain the classical Archimedes law.

Thus, the change in the vortical momentum (2.3) takes place in accordance with laws analogous to the laws of the change in the true momentum (where the latter exists).

In the practically important case of the floating-up of the cavity, Eq. (2.4) is written very simply:

$$dP_i/dt = -g_i V.$$

In the case of the motion of a solid body in a liquid, it is convenient to introduce the total momentum R of the system solid body–liquid:

$$\mathbf{R} = m\mathbf{U} + \frac{1}{2}\left\{\int_G \mathbf{r} \times \boldsymbol{\omega} dV + \int_{\partial K} \mathbf{r} \times (\mathbf{n} \times \mathbf{v})\, dS\right\}, \tag{2.7}$$

where m is the mass of the body; U is the velocity of the body. For R, the following equality from (2.4) is valid:

$$d\mathbf{R}/dt = \mathbf{g}(m - \rho V),$$

where ρ is the density of the liquid.

In (2.7) the first term is the momentum of the body and the second the apparent momentum for the case of vortical flows. It can be shown that, in the case of potential flow, (2.7) assumes the form

$$\mathbf{R} = (m + \mu)\mathbf{U},$$

where μ is the apparent mass [4]; here

$$\mu\mathbf{U} \equiv \int_{\partial K} \varphi \mathbf{n} dS = \frac{1}{2} \int_{\partial K} \mathbf{r} \times (\mathbf{n} \times \mathbf{v})\, dS$$

(φ is the potential of the flow); i.e., in this case, the momentum $\mathbf{P}$ coincides with the apparent momentum $\mu\mathbf{U}$.

Let us now examine a flow of an incompressible unbounded liquid, with a vorticity concentrated in some finite region of the flow. Then for an arbitrary finite region A, containing the whole vorticity, we have

$$\mathbf{P} = \frac{1}{2}\int \mathbf{r} \times \boldsymbol{\omega} dV = \int_A \mathbf{v} dV + \frac{1}{2}\int_{\partial A} \mathbf{r} \times (\mathbf{n} \times \mathbf{v})\, dS. \tag{2.8}$$

This relationship may be assigned the following meaning: If in the liquid a closed surface can be traced outside of which $\omega = 0$, then the vortical momentum of the flow is represented in the form of the inherent momentum of the given amount of liquid and its additional momentum.

It is well known that with the motion of steady-state vortical rings in an ideal incompressible unbounded liquid, along with a ring there moves a certain liquid volume of unchanged form called the "atmosphere" of the vortical ring.

We replace this volume by a solid body, having a density equal to the density of the liquid, leaving the remaining part of the flow unchanged.

The equality (2.8) enables the following question from [5] to be answered: Will the vortical momentum (2.2) of a vortex ring be equal to the total momentum (2.7) of the body–liquid system?

Taking as A in (2.8) the "atmosphere" of the vortical ring, and using $\mathbf{P} = \mu\mathbf{U}$, as has been noted earlier, the question posed can be answered in the affirmative.

§3. Vortical Momentum of an Inhomogeneous Incompressible Liquid. Let us generalize the definition of the vortical momentum (2.2) for the case of flows of an incompressible unbounded liquid with $\rho \neq$ const; we obtain the expression

$$\mathbf{P} = \frac{1}{2}\int \mathbf{r} \times \operatorname{rot} \rho\mathbf{v} dV. \tag{3.1}$$

The limitations which were imposed on the field of the velocities must now be laid on the field $\rho\mathbf{v}$, with the exception that in (3.1) there is the possibility of a discontinuity of ρ at some liquid surface, not departing to infinity (stratified flows). In the latter case, surface integrals appear in (3.1), since differentiation of the exponential function gives a δ function at the corresponding surfaces. Let the discontinuity of ρ occur at the surface C. Then (3.1) assumes the form

$$\mathbf{P} = \frac{1}{2}\left\{\int \mathbf{r} \times \operatorname{rot} \rho\mathbf{v} dV + \int_C [\rho]\, \mathbf{r} \times (\mathbf{n} \times \mathbf{v})\, dS\right\},$$

where $[\rho]$ is the discontinuity of ρ at C; $\mathbf{n}$ is a normal to C; the first integral is taken over the whole space, including C. This way of writing the momentum is applicable, for example, for vortical rings with a core made of a material different from the surrounding medium.

The correctness of the definition (3.1) is shown by the dynamic equation for P, having the form

$$\frac{d\mathbf{P}}{dt} = -\mathbf{g}\int \{\rho(\mathbf{r}) - \rho(\infty)\} dV,$$

with the asymptotic limitations on

$$|\rho(\mathbf{r}) - \rho(\infty)|\, r^3 \to 0 \quad \text{as} \quad r \to \infty.$$

The liquid can be either viscous or ideal. The proof is carried through analogously to the proof of Theorem 1.

§4. The Moment of Momentum of an Incompressible Liquid. Analogously to the definition of the momentum discussed in the present article, the moment of momentum of the flow outside the bounded region K can be considered, using the definition for the moment of momentum of a liquid filling the whole space [4]:

$$\mathbf{M} = \frac{1}{3}\int \mathbf{r} \times (\mathbf{r} \times \boldsymbol{\omega})\, dV \quad (\rho = 1).$$

To obtain the moment of momentum for the flow outside of K here it is sufficient to substitute the distribution of the vortices, containing the surface distribution of the density $\mathbf{n} \times \mathbf{v}$ at ∂K. Then

$$\mathbf{M} = \frac{1}{3}\left\{\int_G \mathbf{r} \times (\mathbf{r} \times \boldsymbol{\omega})\, dV + \oint_{\partial K} \mathbf{r} \times (\mathbf{r} \times (\mathbf{n} \times \mathbf{v}))\, dS\right\}.$$

The dynamic equation for **M** is written in the form

$$\frac{dM_i}{dt} = -\int_{\partial K} \varepsilon_{ikl} x_k \sigma_{lp} n_p\, dS.$$

This equation is derived analogously to the proof of Theorem 1, with almost the same limitations.

The generalization of the moment of momentum for the case of an inhomogeneous liquid is written in the form

$$\mathbf{M} = \frac{1}{3}\int \mathbf{r} \times (\mathbf{r} \times \operatorname{rot} \rho \mathbf{v})\, dV.$$

The discussion and justification of this definition are analogous to the discussion in Sec. 3.

In conclusion, let us dwell on an evaluation of the limitations adopted on the asymptotic of the field of the velocity

$$|\mathbf{v}(\mathbf{r}, t)|r \to 0 \quad \text{as} \quad r \to \infty.$$

Such a limitation is natural for flows of an ideal liquid with an asymptotic vorticity $|\boldsymbol{\omega}(\mathbf{r}, t)| < c_1(t)/r^{4+\varepsilon}$ $(\varepsilon > 0)$ and allows of the presence of sources in the bounded region, i.e., a change in the volume of the region K.

In the case of a viscous liquid, such limitations are not obvious. Thus, with steady-state flow of a viscous liquid around a body, behind the body there is a wake region, in which $|\mathbf{v}| \sim c_2/r$ with $r \to \infty$ [6]; i.e., the postulation is not satisfied; c_1 and c_2 are arbitrary bounded functions of the time, not depending on the coordinates.

However, on the other hand, the concept of vortical momentum cannot be applied to the fully established flow of a viscous liquid around a solid body, since a finite force, acting on the liquid for the course of an infinite "establishment time," must lead, in accordance with (2.4), to an infinite vortical momentum of the liquid. Thus, in the case of a viscous liquid, the above discussion is valid only for finite intervals of time, from the moment of the start of the motion.

The author wishes to thank B. A. Lugovtsov for posing these questions.

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